Singularity theory study of overdetermination in models for L–H transitions

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Two dynamical models that have been proposed to describe transitions between low and high confinement states (L–H transitions) in confined plasmas are analysed using singularity theory and stability theory. It is shown that the stationary-state bifurcation sets have qualitative properties identical to standard normal forms for the pitchfork and transcritical bifurcations. The analysis yields the *codimension* of the highest-order singularities, from which we find that the unperturbed systems are overdetermined bifurcation problems and derive appropriate *universal unfoldings*. Questions of mutual equivalence and the character of the state transitions are addressed.

It is a well-known fact that an overdetermined system of equations usually has no exact solutions. In this paper we report a novel application of singularity theory methods [1] to resolve a subtle case of overdetermination in two dynamical systems that model L(low)–H(high) confinement state transitions and associated edge-localized modes (ELMs) in confined plasma devices [2,3]. The analysis also addresses the much-discussed question of whether second-order state, or "phase", transitions occur in these systems. Since both models are based on sound physics and seek to describe the same phenomena, we discuss briefly the issue of equivalence, in terms of the singularity theory results.

The semiotics and dissemination of singularity theory owe much to the elementary catastrophe theory proposed by Thom [4]. In substance, however, the provenance of singularity theory can be traced to the work of Poincaré, and the original exposition was by Whitney [5]. It was subsequently developed rigorously and extended by many others, e.g. [1,6,7]. Successful applications have included diverse problems in mechanical, biological, and chemical [8,9] systems. This is the first systematic application to bifurcation problems in plasma physics.

In the singularity theory approach, the qualitative properties of a dynamical system are characterized by classifying the singularities in the set of stationary states, or *bifurcation diagram*, over the parameter space. In the bifurcation diagram of an idealized dynamical model, a degenerate singular point that is *persistent* to variations of the parameters may be a symptom that the model is overdetermined in a way that is not obvious to cursory inspection. The singular point is defined by the bifurcation problem — the stationary-state equation of the dynamical system — *plus* equations for the zeros of certain derivatives of the bifurcation problem. This augmented system may have more equations than unknowns because one or more terms incorporating additional parameters are missing. In the language of singularity theory [1], *the codimension* (see below for definition) *of a persistent, degenerate singular point exceeds the number of auxiliary parameters.* An idealized model containing this type of point cannot exhibit the qualitative features of a more realistic model where perturbational terms *unfold* the singularity. What is worse is that real-world experiments, where perturbations are inevitably present, are likely to exhibit behavior that cannot be predicted by such a model.

The two models investigated here describe L- and H-mode dynamics and ELMs in a unified manner, and were derived independently by Sugama and Horton [2] (SH) and Lebedev *et al* [3] (LDGC). Both models describe the coupled evolution of state variables related to the pressure gradient, the shear of the poloidal flow, and the level of magnetohydrodynamic fluctuations in the edge region of a tokamak. Since we are concerned mainly with the stationary states we do not reproduce the dynamical equations, although it should be kept in mind that the stability analysis (which is summarized in the bifurcation diagrams) necessarily refers to the dynamics. In this paper we show that a canonical analysis of bifurcations innate to these systems as given provides *internal* evidence that the derivations may have neglected important physics.

The singularity theory analysis essentially consists of three steps. (1) Each model is formulated as the steady-state, scalar bifurcation problem $g(x, \lambda) = 0$, where x is the chosen state variable and λ is the chosen control parameter. The bifurcation diagrams are found to contain one or more *persistent* degenerate singularities. (2) We show that the g are (locally) strongly equivalent to simple, generic normal forms h. This solves the following *recognition problem:* what conditions must a given g satisfy in order to evince qualitative equivalence to a given normal form h? Concomitantly, we obtain two valuable pieces of information: the character of the most degenerate singularity in each model and the *codimension* of this singularity, defined by the minimum number k of independently variable auxiliary parameters required to net all possible qualitative behaviors and obtain a *universal unfolding*. (3) The bifurcation sets are perturbed to obtain universal unfoldings of the form $G(x, \lambda, \alpha_1, \ldots, \alpha_k) = 0$, where the k auxiliary or unfolding parameters $\alpha_1, \ldots, \alpha_k$ are non-redundant and all other unfoldings of g may be extracted from G. Singularity theory

is concerned with the qualities of steady-state bifurcation problems that determine the dynamics of an associated physical system. The key concepts of codimension and qualitative equivalence, together with the universal unfoldings and stability considerations, allow us to construct a complete catalogue of the bifurcation behavior.

The SH model: This may be expressed as the dimensionless bifurcation equation

$$\begin{aligned} g(u,q,d_a) &= \left(qd_a u^{-2} - 1\right)\left(-q + um\left(u\right)\right), \\ m(u) &= u^p(b + au^{1-p}), \end{aligned} \tag{1}$$

obtained by eliminating in the steady state the two other dynamical variables f and k in favor of $u \propto$ the potential energy of the pressure gradient. The control parameter q is the power input, d_a is the reciprocal of the anomalous diffusivity, and $m(u)$ is the anomalous viscosity. In Sugama and Horton's numerical work d_a was set to 1, p was given values of $-3/2$ (case A) and -1 (case B), and a and b were given as positive numerical factors. (*Note:* The dynamical equations also contain a parameter c which cancels from Eq. (1).) Figure 1 shows the bifurcation diagrams for case A and case B. (In all diagrams stable solutions are indicated by solid lines, unstable solutions by dashed lines, and branches of limit cycles by dotted lines marking the maximum and minimum amplitude.) It was assumed that the transition from the lower stable solution branch (L-mode) to the upper stable branch (H-mode) must occur at the singular point A, where the steady-state shear flow kinetic energy $f = \left(u^2 - d_a q\right)/cu$ becomes unphysical. The transition is discontinuous for case A and continuous for case B. The H-mode branch becomes unstable at a Hopf bifurcation [1] to stable limit cycles, identified as ELMs. The SH model thus predicts hysteresis of the L–H transition and oscillating and quiescent H-modes, which accords with recent experimental observations [10–12]. However, the derivative discontinuity at A is problematic. For case A the transition was described as first-order, but it occurs at what appears to be a highly degenerate point. For case B the transition was described as second-order. It should also be noted that the singular point A is *persistent* to variations in d_a, a, and b. For these reasons we suspect that there may not be enough independent parameters in the model. Solution of the recognition problem, step (2), indicates that the model may be overdetermined as a bifurcation problem.

Proposition 1.—Equation (1) with $d_a = d_{a0}$, $p < -1$ is a germ that is strongly equivalent to the normal form

$$h(x,\lambda) = -x^3 + \lambda x. \tag{2}$$

(The term "germ" is explained as follows: two functions $g_1(x,\lambda)$ and $g_2(x,\lambda)$ are equal as germs if they coincide on some neighborhood of a fixed point x_0, λ_0.) *Proof.*—We apply the following theorem, adapted from [1]: *Theorem.—A germ $g(x,\lambda)$ is strongly equivalent to the normal form $h(x,\lambda) = \varepsilon x^3 + \delta\lambda x$ if and only if, at the fixed point (x_0,λ_0),*

$$g = g_x = g_{xx} = g_\lambda = 0, \;\; g_{xxx} \neq 0, \;\; g_{\lambda x} \neq 0 \tag{3}$$

where $\varepsilon = \operatorname{sgn} g_{xxx}$, $\delta = \operatorname{sgn} g_{\lambda x}$. In Eq. (1) we identify the state variable $u \equiv x$ and the distinguished parameter $q \equiv \lambda$ and evaluate the defining and non-degeneracy conditions (3) at the point $A = (u_0, q_0)$. We find that $g = g_u = g_q = 0$, $g_{uu} = 4a\left(-1+p\right) - 4\left(1+p\right)/d_a = 0$ for $d_a = d_{a0} = \left(1+p\right)/a\left(-1+p\right)$, and $g_{uuu} = 12\left(1+p\right)/u_0 d_{a0}$, $g_{uq} = 2/u_0$. Equation (2) for the normal form is inferred. It is the prototypic pitchfork [1], a codimension 2 bifurcation which requires *two* auxiliary parameters for an unfolding that contains, to qualitative equivalence, all possible perturbations of g. We see that the defining conditions for the point A yield a system of *four* algebraic equations in what is effectively only *two* variables — u and q. To resolve the overdetermination we propose a universal unfolding of Eq. (1).

Proposition 2.—The bifurcation function

$$G(u,q,d_a,\alpha) = g(u,q,d_a) + \alpha \tag{4}$$

is a universal unfolding of the germ (1) for $p < -1$. It is equivalent to the prototypic universal unfolding of the pitchfork $G(x,\lambda,\alpha,\beta) = -x^3 + \beta x^2 + \lambda x + \alpha$, where $d_a = d_{a0} \pm \beta$. The proof is not presented here; instead we focus on the qualitative consequences. (The physical interpretation of the unfolding parameter α is discussed below.) Specifically, Eq. (4) encapsulates the generic behavior of the SH system. The four *qualitatively* distinct bifurcation diagrams are shown in Fig. 2(a)–(d), of which (a) and (b) are physically relevant because $\alpha < 0$ leads to dynamical violation of the condition $f \geq 0$. In (a) the L–H and H–L transitions occur at non-degenerate limit points. No marked transition to H-mode occurs at all in (b). Now it can be seen why the unperturbed bifurcation set, Fig. 2(e), and the partially perturbed bifurcation set, Fig. 1(a), cannot predict the results of experiments. *The singularity that exists in these sets (point A) is not even present when α is nonzero.* We also see that changes in the auxiliary parameters around the critical values can lead to incomparably different bifurcation behavior.

What of case B? *Proposition 3.—Equation (1) with $p = -1$ is a germ that is strongly equivalent to the normal form $h(x,\lambda) = -x^2 + \lambda^2$, a codimension 1 bifurcation known as the transcritical bifurcation.* *Proof.*—We apply the

following theorem from [1]: *Theorem 2.—A germ $g(x,\lambda)$ is strongly equivalent to the normal form $h(x,\lambda) = \varepsilon\left(x^2 - \lambda^2\right)$ if and only if, at the fixed point (x_0, λ_0),*

$$g = g_x = g_\lambda = 0,\ g_{xx} \neq 0,\ \det\begin{pmatrix} g_{xx} & g_{\lambda x} \\ g_{\lambda x} & g_{\lambda\lambda} \end{pmatrix} < 0 \tag{5}$$

where $\varepsilon = \mathrm{sgn} g_{xx}$. These conditions in Eq. (1) yield $g = g_u = g_q = 0$, $g_{uu} = -8a$, $\det d^2 g = -4\left(a d_a - 1\right)^2 / u^2$. Equation (4) in this special, fragile case is a one-parameter universal unfolding, indifferent to the value of d_a. It yields *two* qualitatively distinct bifurcation diagrams, shown in Fig. 3. Note that the bifurcation structure here excludes the possibility of hysteresis.

The LDGC model: The steady states are summarized in the bifurcation diagram of Fig. 4, where the control parameter ϕ is the particle flux and p is the pressure gradient. The lower stable branch is identified as L-mode. At A the transition to the intermediate stable branch AB, identified as H-mode, is described by Lebedev *et al* as analogous to a second-order phase transition. At B the system moves onto the $p = 1$ branch in another continuous transition, but is said to remain in H-mode. The first Hopf bifurcation initiates a branch of *unstable* limit cycles and the second terminates a branch of *stable* limit cycles, identified as ELMs. The point C is the intersection of the $p = 1$ branch and the unstable AC branch. Near B and C the bifurcation equations may be written, respectively, as

$$g_B(p,\phi) = \gamma\left(\phi - \tilde{d}\mu p\right)(p-1)\Big/ p\left(\tilde{d} - \tilde{d}_m\right), \quad \text{and} \tag{6}$$

$$g_C(p,\phi) = \gamma\left(p^2\tilde{d} - \phi\right)(p-1)\Big/ p\tilde{d}_m. \tag{7}$$

As before, we use the singularity theory analysis to focus on qualitative structure. Using theorem 2 we find that at points B and C there is a transcritical bifurcation, which requires the single auxiliary parameter α' for a universal unfolding. The two qualitatively distinct bifurcation diagrams are shown in Fig. 5. In (a) a branch of stable limit cycles connects the two stable stationary branches. In (b) the branches of stable stationary solutions are unconnected. The structure of the limit cycle branch implies that (on a phase plane) a stable orbit is surrounded by an unstable orbit. The point A in Fig. 4 clearly is not unfolded by the one-parameter perturbation. Somewhat surprisingly, it is the limit point of the branch of the branch CAB, which actually coincides along AB with the continuous branch $0AB$. (This result is detailed elsewhere.) A limit point is its own universal unfolding, i.e., persistent to small perturbations.

In summary: (1) The SH model in general is a codimension 2 bifurcation problem, containing a pitchfork, that requires two unfolding parameters for a universal unfolding and hence complete determination. The critical values of the unfolding parameters α and d_a are respectively 0 and $(1+p)/a(-1+p)$, $p < -1$. (2) The LDGC model is a codimension 1 bifurcation problem, containing two transcritical bifurcations. A universal unfolding is provided by a single auxiliary parameter α'. The two models are therefore *structurally dissimilar* in general form. However, the fact that they describe the same phenomena suggests that the LDGC model may be a partially collapsed codimension 2 system, and in a forthcoming work we show that this is indeed the case. *A fortiori* we can also say that second-order phase transitions in these systems, if they exist, could only be observed on variation of at least two parameters simultaneously. In many bifurcation problems pitchforks occur in the presence of $\mathbf{Z}_2$ equivariance in the governing equations for the system, that manifests as a physically invariant property. (A function $\phi(x)$ has $\mathbf{Z}_2$ symmetry if $\phi(-x) = -\phi(x)$.) A symmetry arises in the dynamical equations for the SH model because the shear of the poloidal flow v' is invariant under the transformation $v' \to -v'$. The unfolding parameter α can therefore be interpreted as a symmetry-breaking term, representing an intrinsic energy (or angular momentum) generation rate that occurs even in a pressure gradient of zero. The $\mathbf{Z}_2$ invariance of the flow shear is not evident in the bifurcation structure of the LDCG model, and α' represents a perturbation of the MHD turbulence level.

Other models for L–H transitions that have multiple solutions include those where the flow shear is due to ion-orbit losses on the plasma edge [13,14] or magnetic field ripple induced particle flux in the core [15]. We feel that singularity theory could play an important role in developing and unifying these models and elucidating the physics of L–H transitions. There is a reasonable expectation that different models, if they appeal to the same general physics, should belong to the same qualitative universality class even though they may differ quantitatively. A wider question is whether the dynamics of infinite-dimensional systems can be approximated by low-dimensional systems such as these. The practical advantages are obvious, and developments in inertial manifold theory [16] have shown that the long-time-scale behavior of infinite-dimensional dissipative systems can occur in a defined finite-dimensional subspace.

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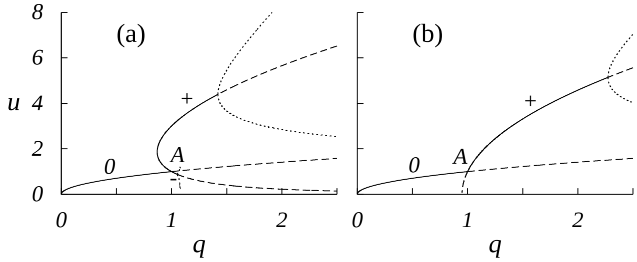

FIG. 1. Bifurcation diagrams of the original SH model. $d_a = 1$, $a = 0.05$, $b = 0.95$, $c = 5$. (a) case A: $p = -3/2$, (b) case B: $p = -1$. Labels indicate the sign of the shear flow energy, thus a minus sign means that the branch is unphysical.

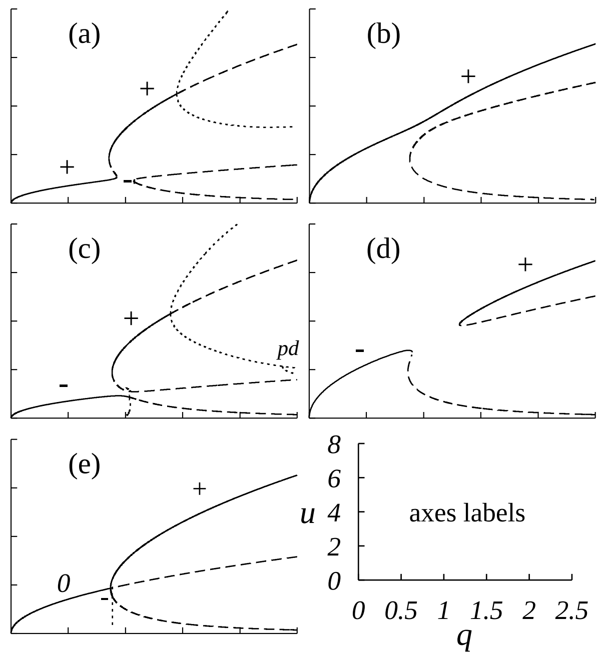

FIG. 2. Bifurcation catalogue for the general SH model, $p < -1$. (a)–(d) The perturbed diagrams, (a) $\alpha = 0.01$, $d_a = 1$, (b) $\alpha = 0.01$, $d_a = 10$, (c) $\alpha = -0.01$, $d_a = 1$, (d) $\alpha = -0.01$, $d_a = 10$. (e) The unperturbed diagram, $d_a = 4$, $\alpha = 0$. $a = 0.05$, $b = 0.95$, $c = 5$, $p = -3/2$. (In (b), (d), and (e) the upper Hopf bifurcation is off-scale.) pd: period-doubling bifurcation.

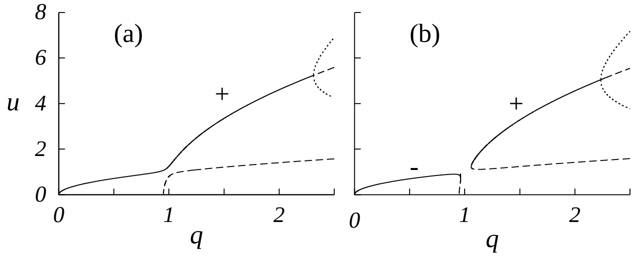

FIG. 3. Bifurcation diagrams for the perturbed SH model, case B. $p = -1$, $d_a = 1$, $a = 0.05$, $b = 0.95$, $c = 5$. (a) $\alpha = 0.01$, (b) $\alpha = -0.01$.

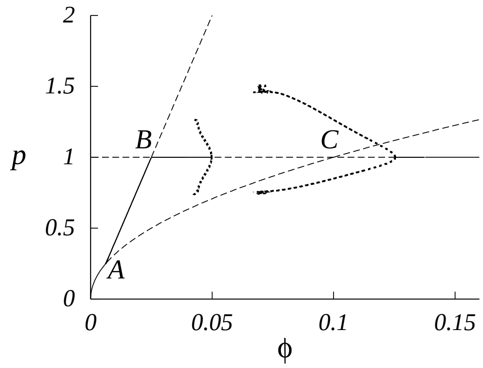

FIG. 4. Bifurcation diagram for the original LDGC model. $\tilde{d} = 0.1$, $\tilde{d}_m = 0.05$, $\mu = 0.25$, $\gamma = 5$.

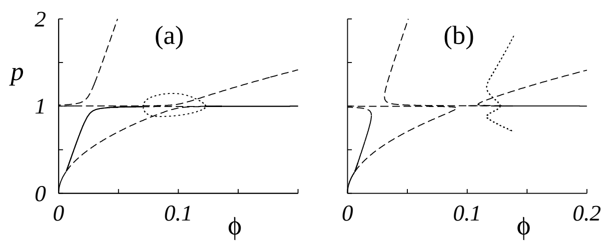

FIG. 5. The two bifurcation diagrams for the universal unfolding of the LDGC model. (a) $\alpha' = 0.01$, (b) $\alpha' = -0.01$. Other parameters as for Fig. 4.